Beyond highway dimension : small distance labels using tree skeletons

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Context : very fast shortest path computation

Recent progress, in particular for transportation networks.

Two ingredients :

- Tool : precompute small "hub sets".
- Graph property : small "hub sets" do exist, e.g. in road networks.

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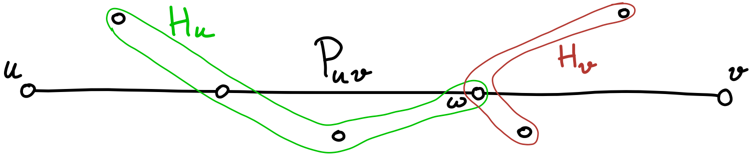
Given a graph G, assign a hub set $H_u \subseteq V(G)$ to each node u, s.t. for all u, v there exists $w \in H_u \cap H_v$ with $w \in P_{uv}$.



Application : Distance labels : $L_u = \{(w, d(u, w)) : w \in H_u\}$

Introduced by [Gavoille et al. '04 ; Cohen et al. 2003], applied to road networks [Abraham et al. 2010-2013], and other practical networks [Akiba et al. 2013]. Approximability results : [Babenko et al. 2013, Angelidakis et al. 2017].

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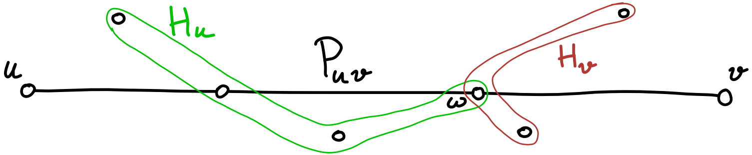

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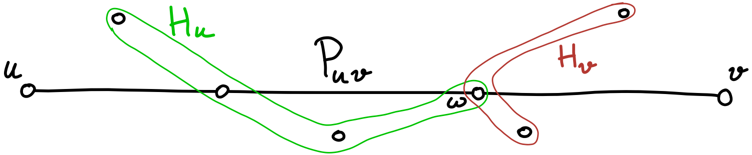

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This talk : More generally, small skeleton dimension !

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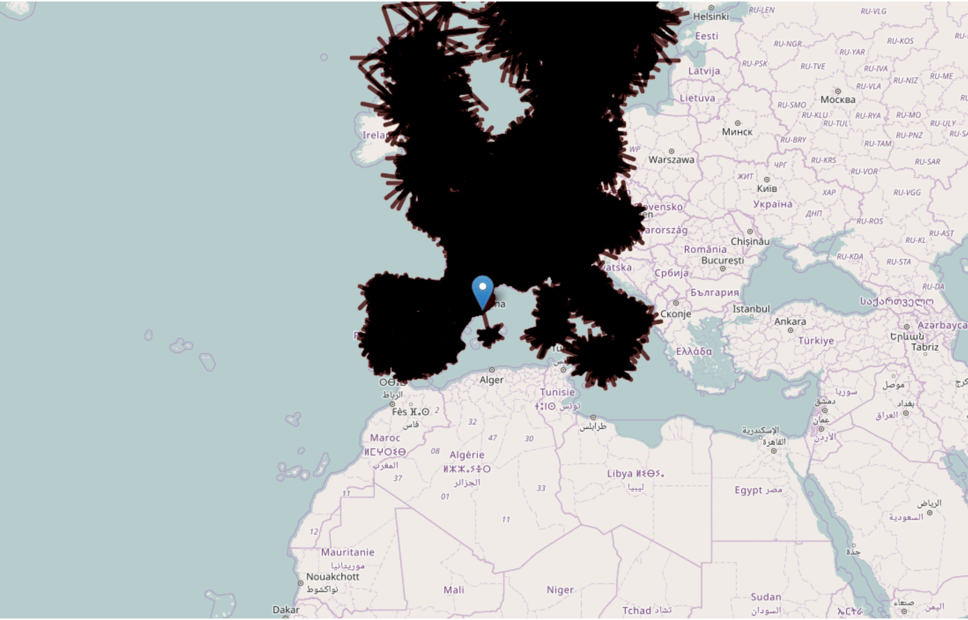
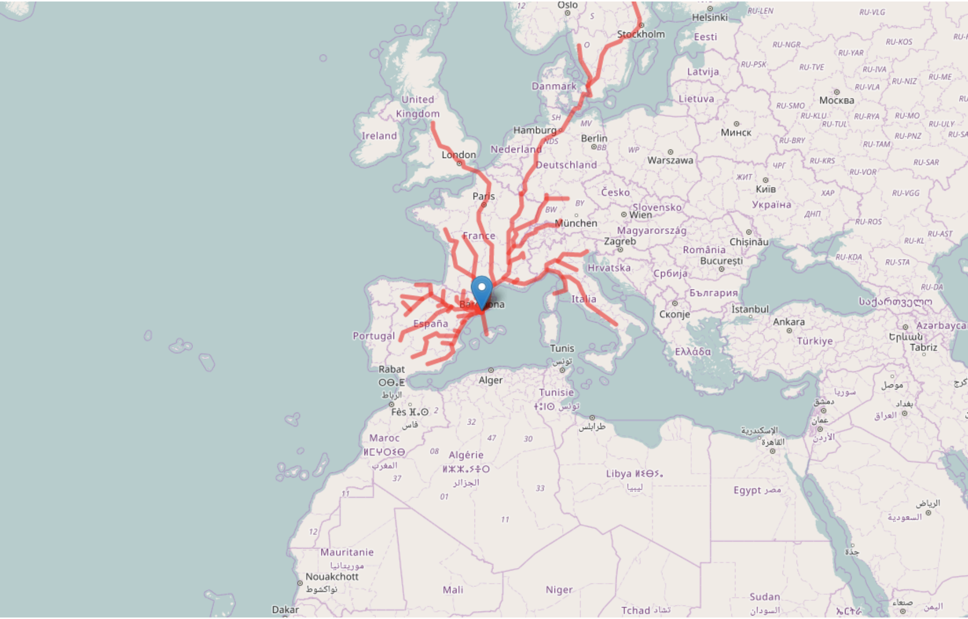
Skeleton dimension

The skeleton dimension k of G is the maximum "width" of a "pruned" shortest path tree.

Barcelona shortest path tree



Barcelona tree skeleton : prune last third



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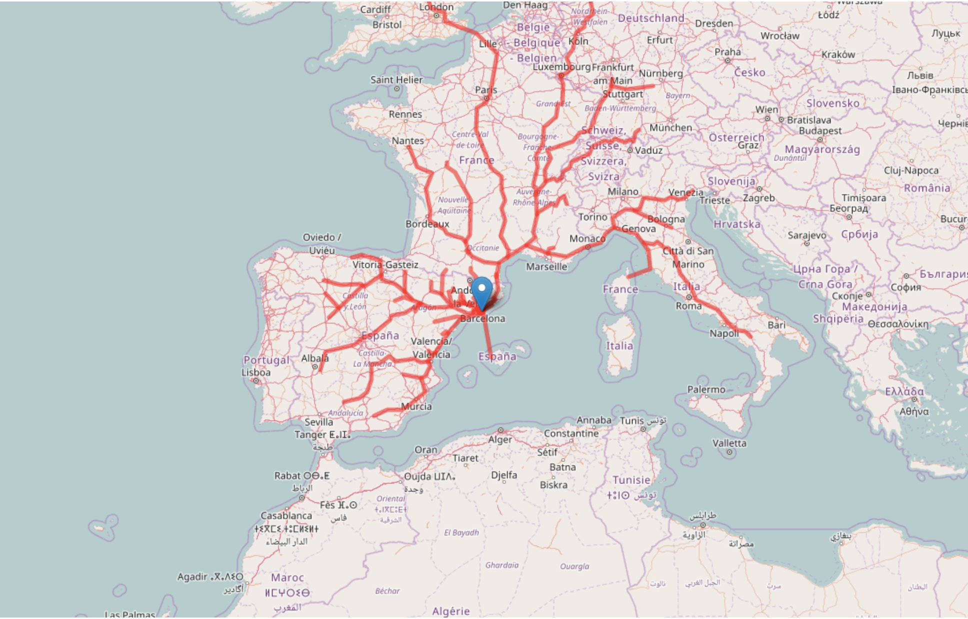

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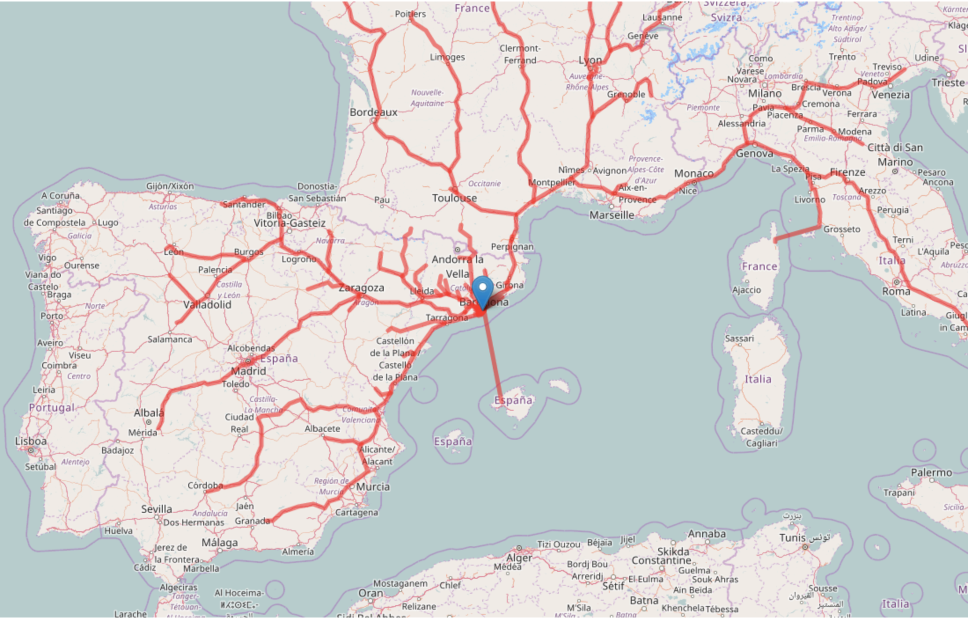

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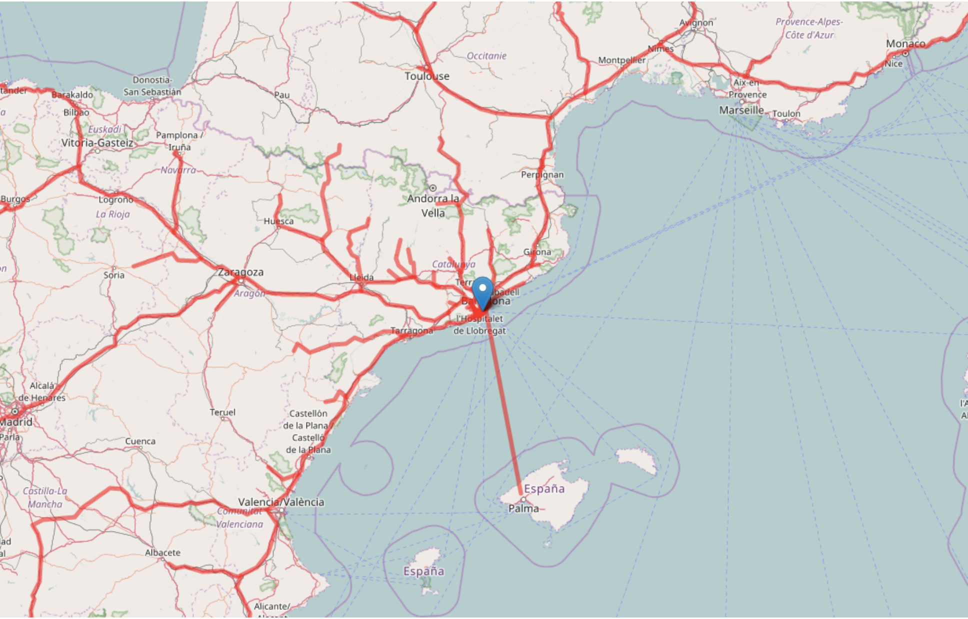


Assumptions

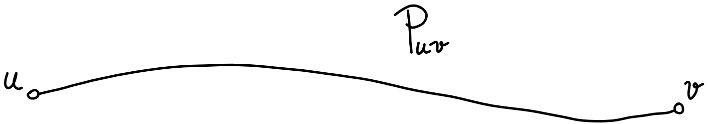
A directed graph G with

unique shortest paths

and integer edge lengths (aspect ratio is $O(D)$).

In the presentation : unweighted undirected graph G.

Tree skeleton



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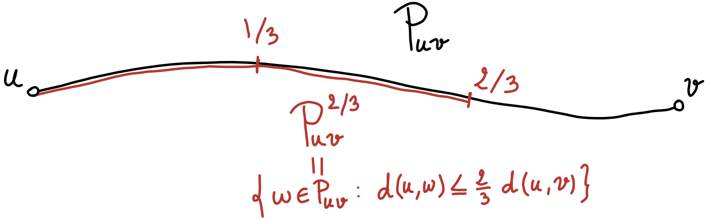

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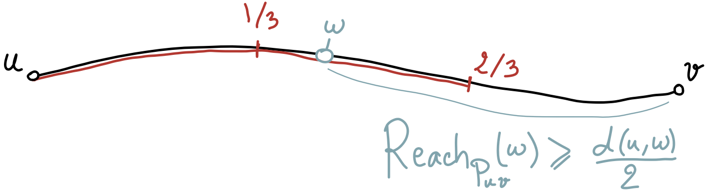

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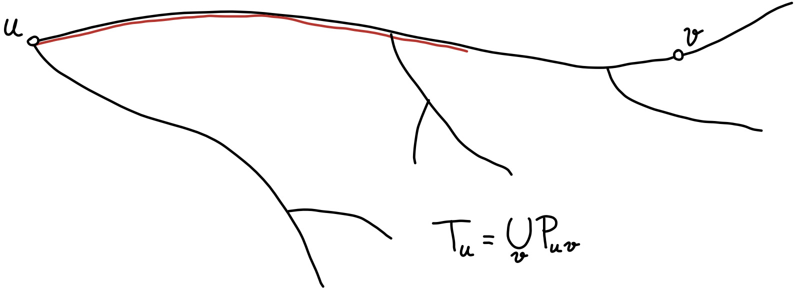

$$T_u = \bigcup_v P_{uv}$$

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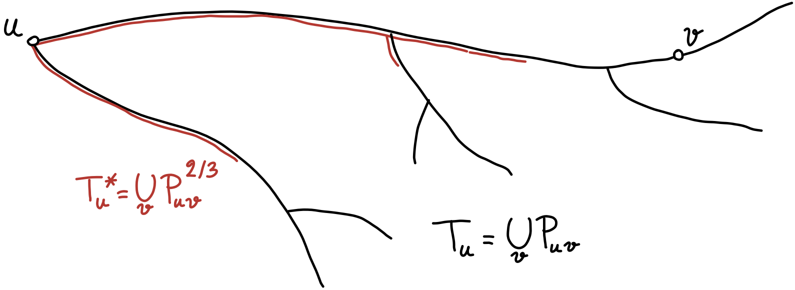

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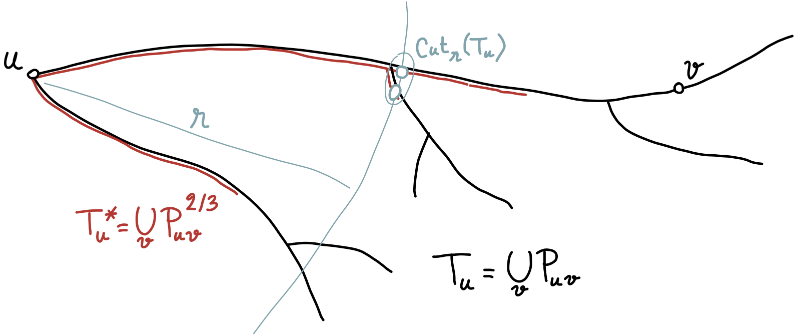

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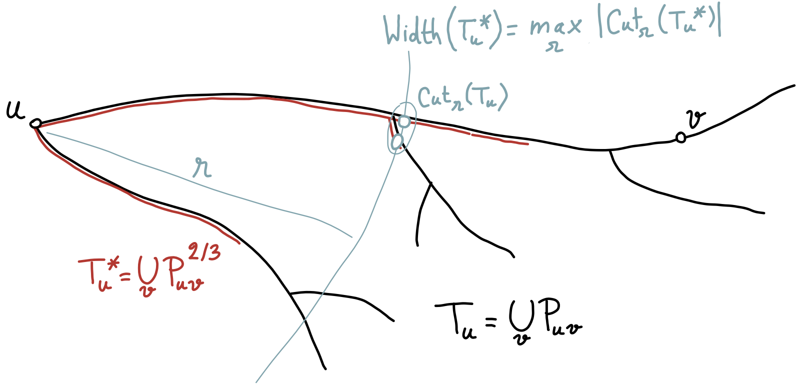

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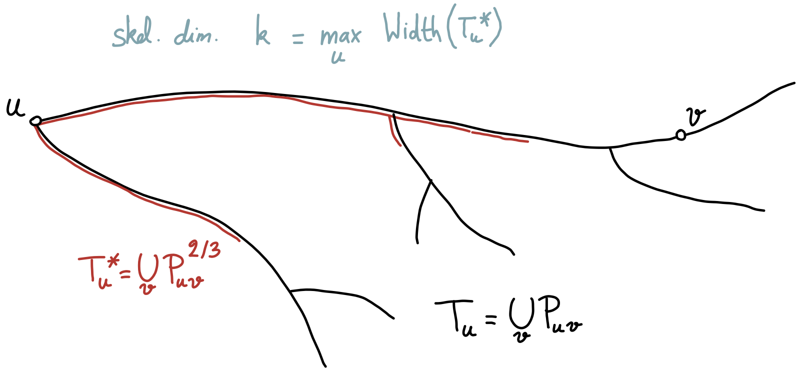

Main result

Theorem

Given a graph G with skeleton dimension k and diameter D, a simple random sampling technique allows to find in polynomial time hub sets with size $O(k \log D)$ on average and maximum size $O(k \log \log k \log D)$ with high probability.

Comparision with highway dimension h :

- more general : $k \leq h$
  (some graphs have $h = \Omega(\sqrt{n})$ and $k = O(\log n)$),
- shorter : $O(k \log \log k \log D)$ vs $O(h \log h \log D)$
  (for polynomial time construction),
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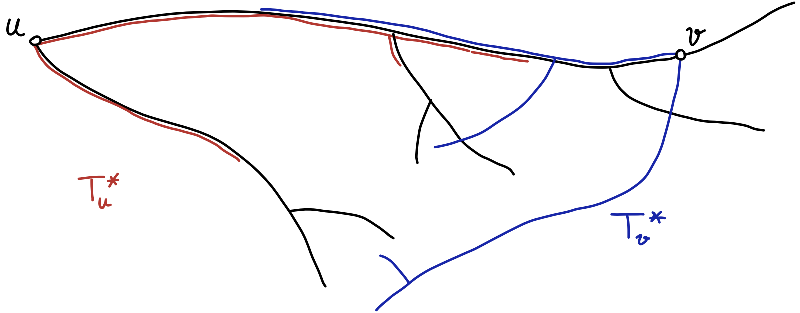

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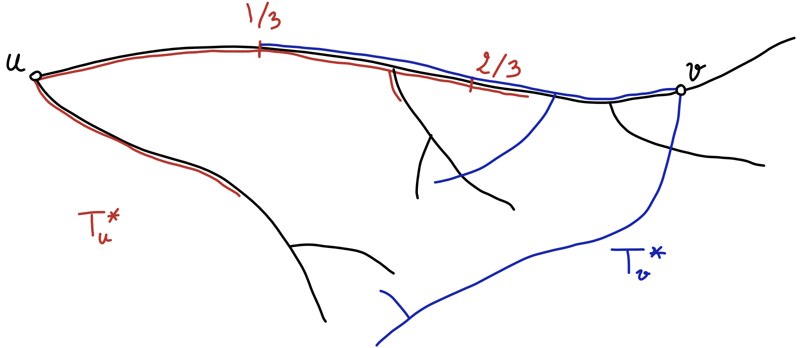

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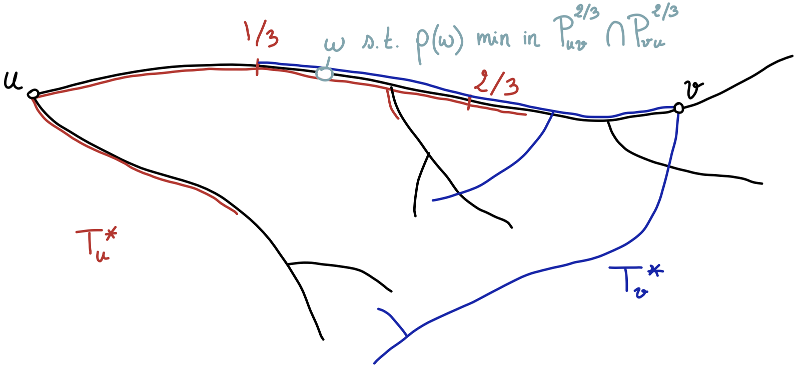

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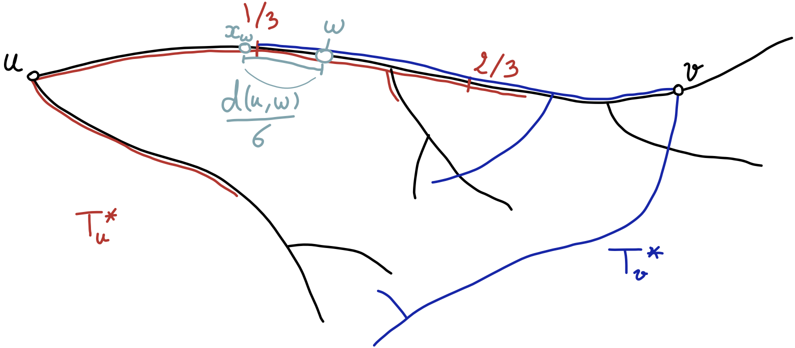

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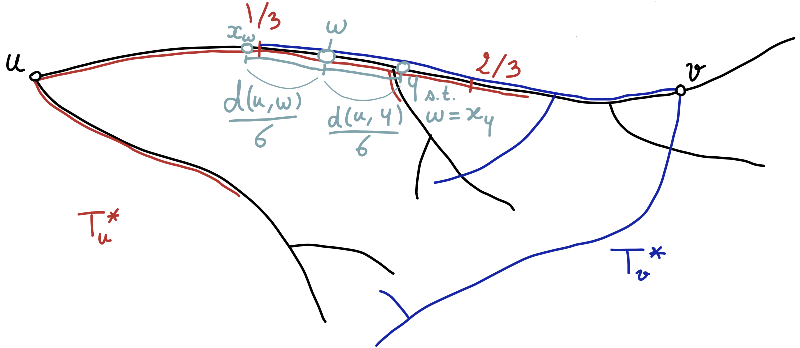

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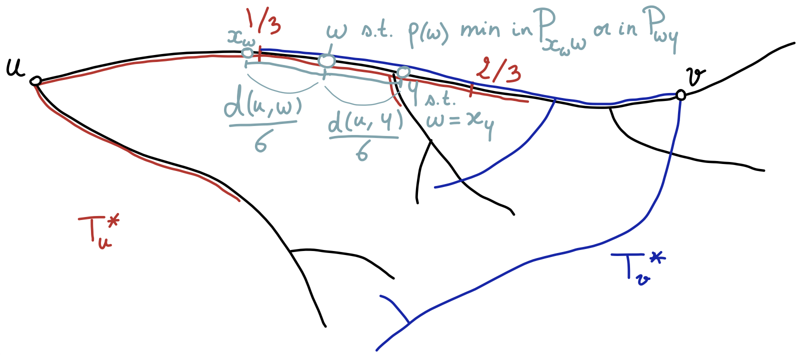

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Draw $\rho(w) \in [0,1]$ u.a.r. for all $w \in V(G)$.

$H_u = \{w \mid \rho(w) \text{ min. in } P_{x_w w}\} \cup \{x_y \mid \rho(x_y) \text{ min. in } P_{x_y y}\}$

(Can be computed in $\widetilde{O}(n+m)$ separately for each node with shared randomness.)

A sub-path $P_{x_y y}$ has length $\frac{d(u,y)}{6}$ and generates a hub in H_u with probability at most $\frac{12}{d(u,y)}$.

$E[|H_u|] \leq \sum_{y \in V(T_u^*)} \frac{12}{d(u,y)} \leq \sum_r |Cut_r(T_u^*)| \frac{12}{r} = O(k \log D)$

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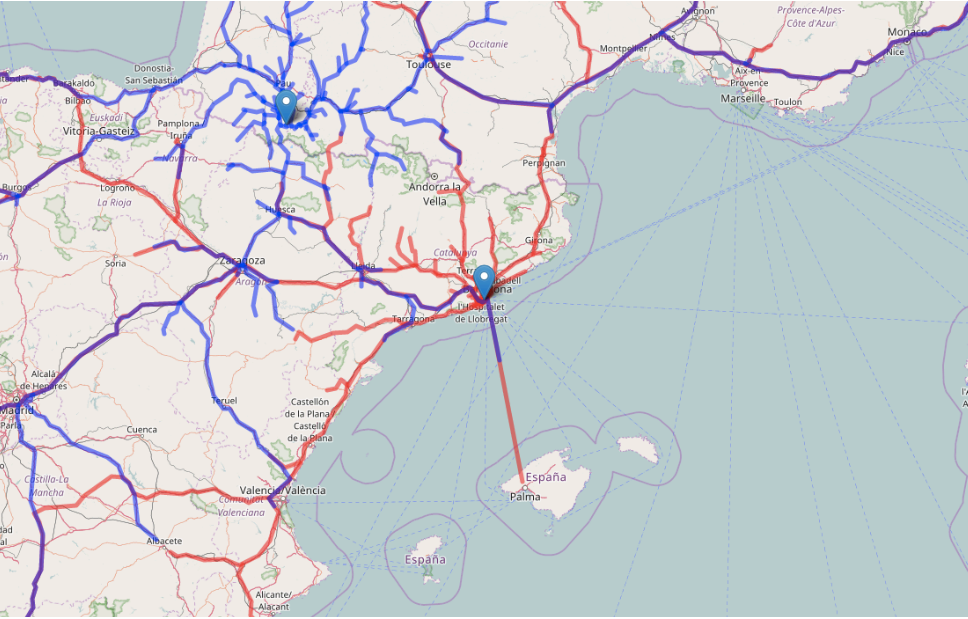
Road networks : two tree skeletons



Technicalities

Branching introduces non-trivial correlations between sub-paths.

We construct edge hub sets.

An edge of length ℓ is virtually subdivided into an unweighted path of length 12ℓ.

Naturally extends to directed graphs.

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Highway dimension

A graph G has small highway dimension h if “long” paths in a given region go through “few” transit nodes.

Highway dimension $\geq$ skeleton dimension

$\mathcal{P}_{ur} = \{P \mid |P| > \frac{r}{2} \text{ and } P \cap B(u,r) \neq \emptyset\}$

H hits $\mathcal{P}_{ur}$ if $H \cap P \neq \emptyset$ for all $P \in \mathcal{P}_{ur}$

Highway dim. $h = \max_{ur} \min_{H \text{ hits } \mathcal{P}_{ur}} |H|$

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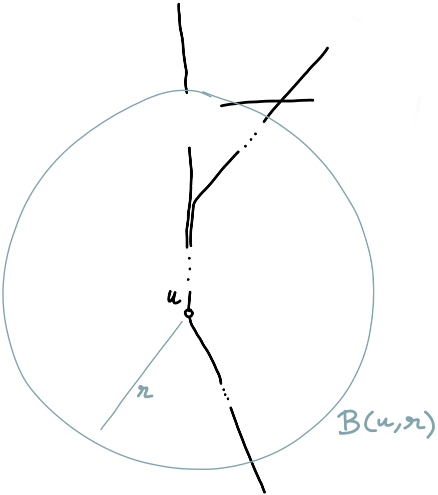

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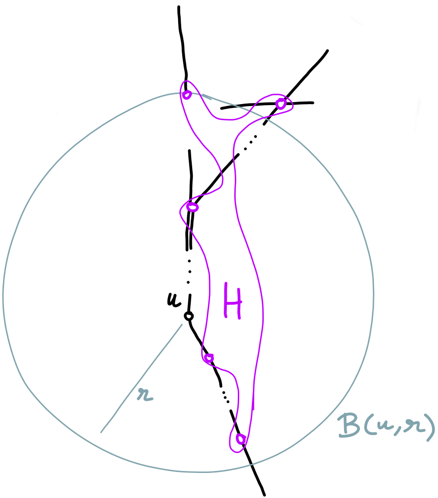

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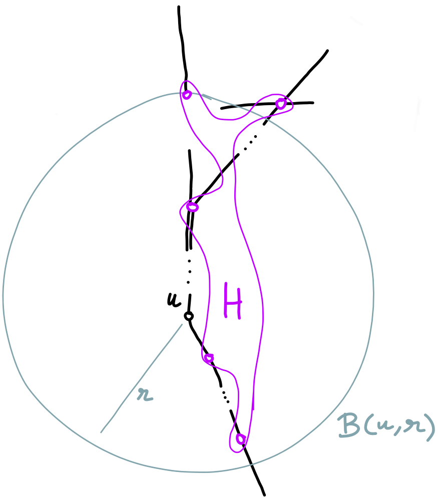

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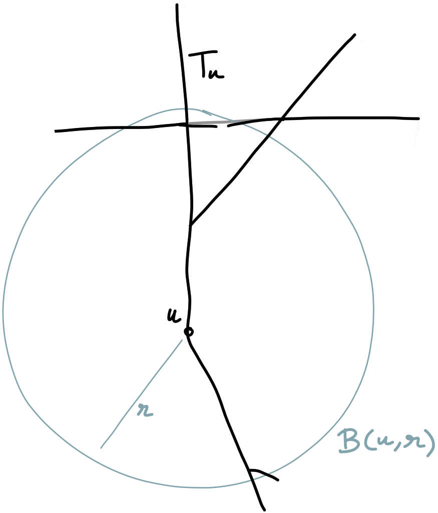

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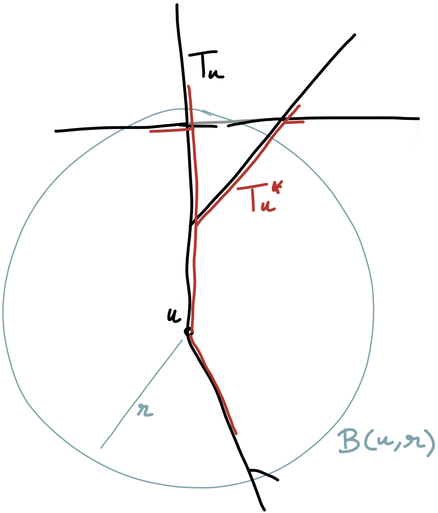

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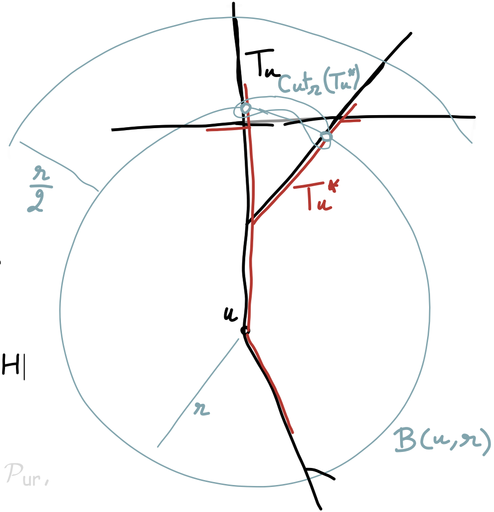

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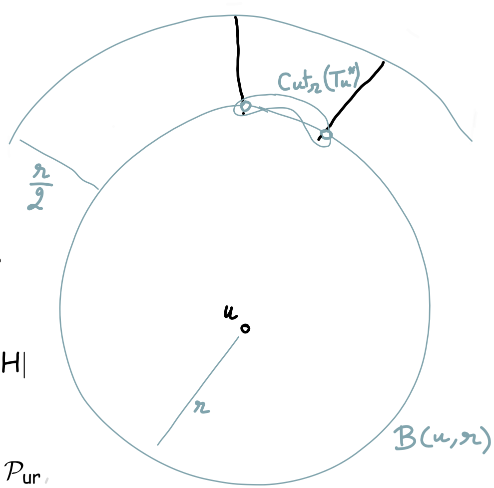

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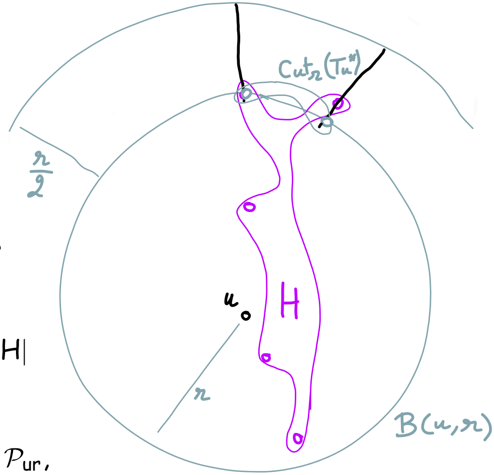

Highway vs skeleton w.r.t. doubling property

A graph G is γ-doubling if any ball $B(u, r)$ can be covered by at most γ balls with radius $\frac{r}{2}$: $\exists H$ s.t. $B(u, r) \subseteq \cup_{v \in H} B(v, \frac{r}{2})$ and $|H| \leq \gamma$.

Proposition

Any graph with highway dimension h and skeleton dimension k is $\min\{h+1, 2k+1\}$-doubling.

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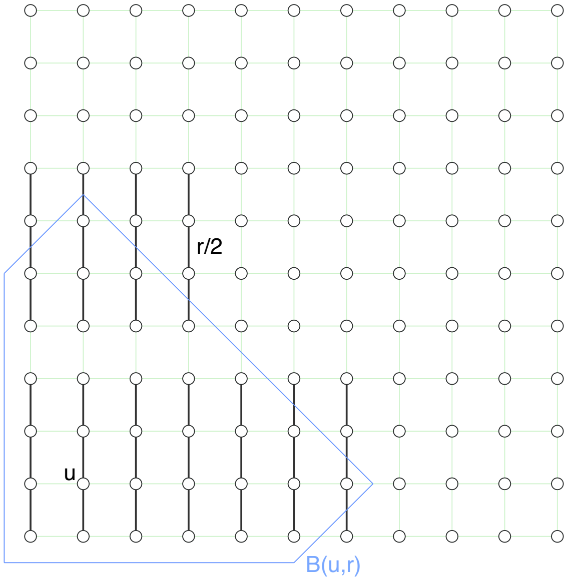

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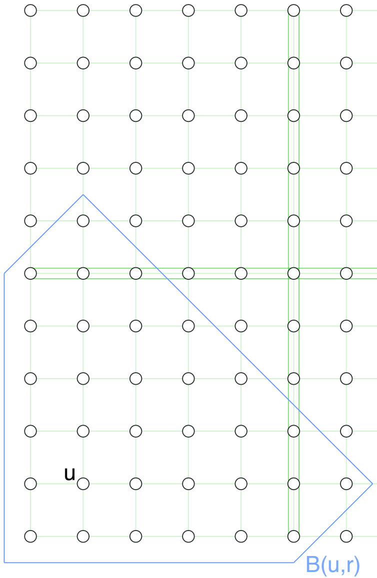

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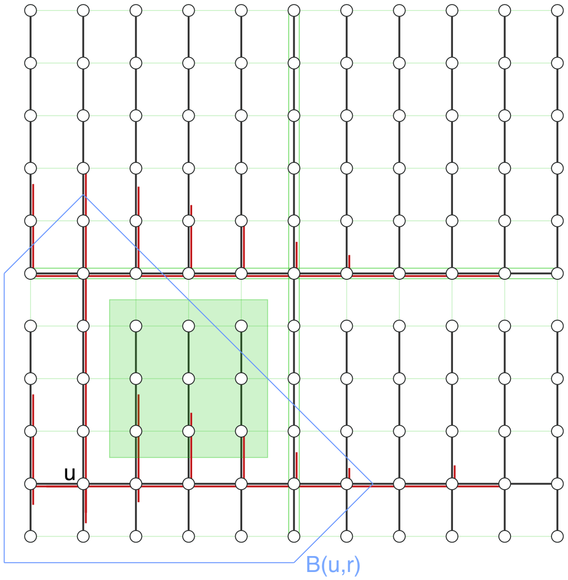

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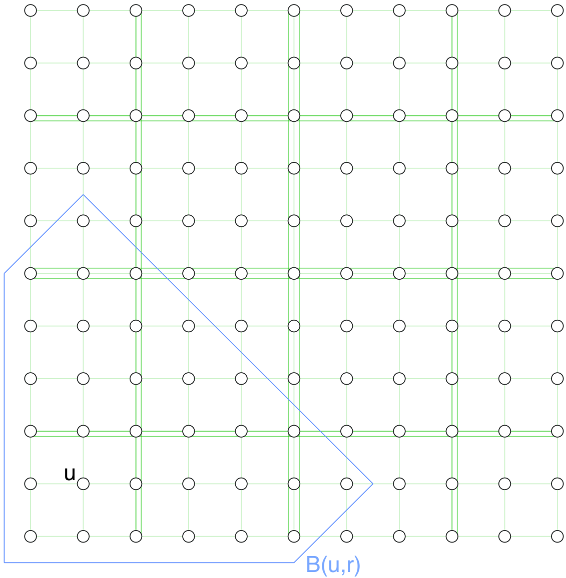
$k = \Theta(\log n)$

Highway vs skeleton in grids



$h = \Theta(\sqrt{n})$

$k = \Theta(\log n)$

Highway vs skeleton in Brooklyn

Packing of 172 paths

Skeleton width 48

Summary

Theorem : Hub sets of size $O(k \log D \max \left\{1, \log \frac{\log n}{\log D}\right\})$ can be constructed in randomized polynomial time for skeleton-dimension-k graphs.

Bonus : improvement of δ-preserving distance labeling in unweighted graphs (building block for $o(n)$ distance labeling in sparse graphs [Alstrup et al. 2010]) :

For $r \geq \delta$, we have $|\mathrm{Cut}_r(T_u^*)| = O(\frac{n}{\delta})$, and thus hub sets of size $O(\log n + \frac{n}{\delta} \log \log n)$.

Perspectives

Other types of transportation networks ?

Skeleton dimension of random spatial networks ? [Aldous 2014]

Beyond skeleton dimension ?

- Small hub sets imply intersecting sub-strees with few leaves.
- Fast computation : additional property (low treewidth, small reach ?).

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Thanks.